

RECOMBINATION OF ELECTRONS IN A PLASMA EXPANDING INTO A VACUUM

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As noted in a paper by one of the authors [1], when a hot ionized gas expands into a vacuum, at a certain moment ionization equilibrium must necessarily break down. Shortly after this point, which may be found by the method indicated in [1], ionizing events become very rare and only recombination occurs in the gas. In [1] photorecombination and triple collisions with the capture of an electron to the ground level of the atom were considered. Here the recombination did not proceed to the end: on expanding to infinity and cooling to zero the gas remained partially ionized.

Papers have recently appeared [2–7] in which the significant role of triple collisions with the capture of electrons to upper atomic levels is noted. The recombination process has a cascade character at low temperatures and densities which are not excessively small. At first, the electron is captured by one of the upper atomic levels in a triple collision with an ion and another electron. Subsequently, as a result of electron collisions of the second kind, and later also as a result of radiative transitions, the bound electron descends through the energy levels to the atomic ground state. The recombination coefficient for such a process depends much more strongly on the electron temperature T than for a triple collision with capture directly by the ground level (as $T^{-9/2}$ as opposed to T^{-1}), and at low temperatures cascade recombination proceeds much more quickly than capture to the ground level. Since this casts doubt upon the conclusions of [1] regarding the residual ionization when a plasma expands into a vacuum, we were led to re-examine the question, which, as will be clear from what follows, is not considerably more complicated.

1. Statement of the problem and initial assumptions. We shall examine a hot ionized monatomic gas expanding with spherical symmetry into a vacuum, in particular, the late, inertial stage of the expansion, when the internal energy has, to a considerable extent, been transformed into kinetic energy, so that the rate of expansion is practically constant and the gas density falls with the passage of time proportionally as $1/t^3$ [8]. We shall consider the change of degree of ionization α with time, for low temperatures after breakdown of ionization equilibrium and the cessation of ionizing processes. Here we assume that the gas contains only singly charged ions.

The recombination rate depends on the electron temperature T . Generally speaking, the electron temperature may be distinguished from the ion (atomic) temperature T_i as a result of the retarded energy exchange between electrons and heavy particles. The temperature change, in its turn, depends on the heat liberated on recombination, since the process is described by a system of kinetic and energy equations in unknown functions of time α , T , and T_i .

Comparison of the photorecombination coefficients [9] and those for recombination by triple collisions [4] shows that for electron densities which are not excessively low, in fact, for

$$N_e > 3.2 \cdot 10^9 (T^\circ / 10^3)^{3.75} \text{ 1/cm}^{-3}$$

the latter is predominant.

We shall assume that this inequality is fulfilled at the moment when the ionizing equilibrium breaks down. It can be shown that in the entire subsequent process neither photorecombination nor recombination in triple collisions with an atom participating as third particle plays a part. We shall assume that the deactivation of a highly excited atom formed by the capture of an electron by an ion proceeds rapidly in comparison with the rate of electron density and temperature variation (in fact, the recombination coefficient in [2–5] is calculated for this condition). In this case the recombination energy is liberated immediately after the capture of an electron by an ion, i.e., the heat release in the gas at a given moment is determined by the recombination rate at that moment $-\alpha d\alpha/dt$.

We shall assume that only that part of the recombination energy (E^* for each event) which is transferred to the electrons in deactivation of excited atoms by collisions of the second kind is converted into heat.

The other portion $I - E^*$ (I is the ionization potential) is initially radiated as spectral line luminescence. Generally speaking, a certain portion of this energy is also converted into heat as time passes (as the result of possible collisional deactivation of an atom excited by resonance radiation). However, we shall assume that the energy $I - E^*$ is totally lost by the gas. The more transparent the gas, the more justified this assumption is. The gas is actually transparent with respect to continuous absorption in the strong expansion stage, which is what we are interested in. The diffusion of resonance radiation lasts to a later stage of the expansion. This effect will not be taken into account in the present paper. We note only that it may lead to an increase in heat release in the gas and to a slowing down of cooling and recombination.

2. Energy equations and equations of recombination kinetics. Let N be the total number of atoms and ions per cubic centimeter and let $\alpha = N_e/N$ be the degree of ionization. The equation for N_e has the form

$$\frac{dN_e}{dt} = -\beta N_e^3 + \left(\frac{dN_e}{dt} \right)_+$$

Here the term $(dN_e/dt)_+ = \alpha dN/dt$ is connected with the expansion of the gas, and, in accordance with [4], the recombination rate constant $\beta = aT^{-9/2} \text{ cm}^6/\text{sec}$, $a = 8.75 \cdot 10^{-27} \text{ cm}^6/\text{sec}^{-1} \text{ ev}^{9/2}$ (we set the special form of coulomb logarithm entering into β equal to unity).

$$\beta = aT^{-9/2} \text{ cm}^6/\text{sec}, \quad a = 8.75 \cdot 10^{-27} \text{ cm}^6 \text{ sec}^{-1} \text{ ev}^{9/2}. \quad (2.1)$$

Passing from the density N_e to the degree of ionization, we obtain

$$\frac{d\alpha}{dt} = -\beta N^2 \alpha^3 = -\frac{aN^2 \alpha^3}{T^{3/2}} \quad (2.2)$$

The energy balance equation for the electron gas, from calculations on a single electron, has the form

$$\frac{d\varepsilon_e}{dt} + \frac{p_e}{\alpha} \frac{dV_1}{dt} = \frac{E^* + \varepsilon_e}{\alpha} \left(-\frac{d\alpha}{dt} \right) - \frac{3}{2} k \frac{T - T_i}{\tau_{ei}} \quad (2.3)$$

Here

$$\varepsilon_e = \frac{3}{2} kT, \quad p_e = N_e kT, \quad V_1 = 1/N,$$

V_1 is the volume per heavy particle, and, in accordance with [10], τ_{ei} is the characteristic time for energy exchange between electrons and ions (exchange with neutral atoms does not play a part).

$$\tau_{ei} = \frac{250 \mu T^{3/2}}{\alpha N \ln \Lambda} \text{ sec},$$

where μ is the ion atomic weight and $\ln \Lambda$ is the coulomb logarithm. Keeping the expansion law $N \sim t^{-3}$, $V_1 \sim t^3$ in mind, we obtain the equation for the electron temperature

$$\frac{dT}{dt} + 2 \frac{T}{t} = \frac{(2E^*/3k) + T}{\alpha} \left(-\frac{d\alpha}{dt} \right) - \frac{T - T_i}{\tau_{ei}} \quad (2.4)$$

In the same way we construct the equation for the ion (atomic) temperature

$$\frac{dT_i}{dt} + 2 \frac{T_i}{t} = \alpha \frac{T - T_i}{\tau_{ei}} \quad (2.5)$$

If the energy exchange between electrons and ions proceeds rapidly, which is true for not very strong expansion, $T \approx T_i$, and equations (2.4) and (2.5) transform to one equation for the over-all temperature,

$$\frac{dT}{dt} + 2 \frac{T}{t} = \frac{(2E^*/3k) + T}{1 + \alpha} \left(-\frac{d\alpha}{dt} \right) \quad (2.6)$$

For very strong expansion, specifically when the inequality $\tau_{ei} > t$ is fulfilled, the exchange slows down and the heavy-particle gas, in which heat release does not take place, cools more rapidly than the electron gas. At this stage $T_i \ll T$ and the third unknown function $T_i(t)$ vanishes from equation (2.4). It can be shown that at this stage the energy transfer to the ions is small, in general, compared to the work of expansion and the heat liberated in recombination, so that in the limiting case of very strong expansion the equation for the electron temperature assumes the form

$$\frac{dT}{dt} + 2 \frac{T}{t} = \frac{(2E^*/3k) + T}{\alpha} \left(-\frac{d\alpha}{dt} \right) \quad (2.7)$$

We shall now investigate how the degree of ionization behaves asymptotically. We write for formal solution of the recombination kinetics equation (2.2)

$$\alpha = \alpha_1 \left[1 + 2\alpha \alpha_1^2 \int_{t_1}^t \frac{dt N^2}{T^{3/2}} \right]^{-1/2} \quad (2.8)$$

(the point t_1 , α_1 , through which the integral curve passes, is determined by the initial conditions [1]). We

seek the asymptotic solution for the electron temperature in the form of a power function of time $T \sim t^{-m}$. If we introduce an analogy with the case of the adiabatic expansion of an ideal gas, when $T \sim N^{(\gamma-1)}$ (γ is the adiabatic exponent), then by the law of expansion $N \sim t^{-3}$ we obtain the following relation of exponents: $m = 3(\gamma - 1)$.

The asymptotic behavior of the degree of ionization α depends on the exponent m . It follows from (2.8) that for $m < 10/9$ the degree of ionization tends in the limit as $t \rightarrow \infty$ to a constant value differing from zero. If $m > 10/9$, then the degree of ionization tends asymptotically to zero according to the power law $\alpha \sim t^{-3/4(m-10/9)}$, and for $m = 10/9$, according to the logarithmic law $\alpha \sim (\ln t)^{-1/2}$, i.e., in the case when $m \geq 10/9$ there is no residual ionization.

Due to the absence of losses of free electron thermal energy, other than that going into the work of expansion (losses by bremsstrahlung are small), the exponent m is bounded by the condition $m \leq 2$ ($m = 2$ corresponds to $\gamma = 5/3$).

Thus, in distinction from [1], recombination does not cease if the electron gas cools down fast enough, regardless of the expansion, and the degree of ionization may decrease without limit. But the cooling law itself depends on the liberation of heat in recombination, and the asymptotic value of the exponent m must be found by solving the system of kinetic and energy equations. In these equations the dependence of E^* on T and N_e remains as yet undetermined. The following two paragraphs are devoted to determining it.

3. Rate of collision deactivation of an excited atom.

In triple collisions the electrons are captured by upper atomic levels with binding energies $E_n \approx kT$. For simplicity, we shall consider the hydrogen atom. At the low temperatures assumed here high quantum numbers correspond to these conditions:

$$n = \sqrt{T/E_n} \gg 1.$$

Here the distance between levels

$$\Delta E_{n, n-1} \approx \left| \frac{dE_n}{dn} \right| = \frac{2E_n}{n}$$

is much less than kT . At first, spontaneous radiative transitions play a minor part, and the energy of a highly excited atom changes as a result of electron collisions.* As is well known [6], collision transitions to the nearest levels are the most probable, transitions "up" and "down" being almost equally probable if $n \geq 1$, $\Delta E_{n, n-1} \leq kT$ (transitions with "jumps" over levels are of relatively very low probability). The change of energy of an optical electron in an excited atom has the character of "diffusion" along the energy axis, and the "diffusion flux" in recombination is directed downwards in the direction of the ground level [3]. Where the separation between energy levels is greater than kT , transitions down are more likely than transitions up, so that the motion along the energy axis is unidirectional in character. The effective boundary between the regions of diffusion and unidirectional

motion is the point on the energy axis where $\Delta E'_{n,n-1} = kT$. The binding energy at this point is equal to

$$E' = 1/2 kT (2I / kT)^{1/2}.$$

The radiation rate increases rapidly as the electron passes to lower levels (see the following paragraph), and after reaching a certain binding energy E^* , "collision" deactivation of the atom gives way to radiation deactivation. Obviously, the quantity E^* represents the mean energy per recombination event, which is transmitted immediately to the free electrons. Thus, in order to evaluate E^* , one must estimate and compare the rates of collision and radiation descent of the electron through the atomic energy levels.

The process of diffusion along the energy axis is described in [3, 4] on the basis of the Fokker-Planck equation (in [3, 4] the energy $E < 0$; here E designates the binding energy, i.e., $|E|$) for a distribution function $f(E, t)$ of bound electrons in phase space:

$$\frac{\partial f}{\partial t} = -A(E) \frac{\partial j}{\partial E}, \quad j = -\frac{D}{A} \left(\frac{\partial f}{\partial E} - \frac{j}{kT} \right), \quad (3.1)$$

$$A(E) = \text{const } E^{1/2}.$$

The quantity D , which represents the diffusion coefficient along the energy axis and is calculated in [4], where it is designated $\langle \Delta E^2 \rangle / 2$, may be represented in the form

$$D \equiv \frac{\langle \Delta E^2 \rangle}{2} = \frac{3}{4} \frac{EkT}{\tau_{ee}}. \quad (3.2)$$

Here $1/\tau_{ee}$ (with an accuracy to the coulomb logarithm) is the frequency of "collisions" between electrons in the plasma**:

$$\frac{1}{\tau_{ee}} = N_e \langle v_e \rangle \pi r_0^2, \quad \frac{1}{\tau_{ee}} = 2.4 N_e (T^0)^{-3/2} \text{ sec}^{-1}, \quad r_0 = \frac{e^2}{3/2 kT},$$

$$\langle v_e \rangle = \left(\frac{8kT}{\pi m_e} \right)^{1/2}. \quad (3.3)$$

The stationary process is considered ($\partial f / \partial t = 0$) in [3, 4] and the flux $j = \text{const}$ is calculated, which also gives the recombination coefficient. We shall consider the nonstationary case in order to determine the law of motion of the electron along the energy axis. We multiply equation (2.1) by E and integrate over all

*Conditions where a very highly excited atom radiates from its first moment of formation are not realized in practice. The reason for this is the fact that under such conditions photorecombination would predominate over recombination in triple collisions. But in photorecombination the electrons are captured for the most part by lower levels.

**The coulomb logarithm, which differs from the quantity entering into the expression for the collision frequency [10], is present in the diffusion coefficient [4]. Here it is set equal to unity.

phase space Γ . Here we bear in mind that in accordance with [3]

$$A(E) \int_{E=\text{const}} d\Gamma = dE.$$

We set the boundary condition $f = 0, j = 0$ for $E \gg kT$ to correspond to the consideration of nonstationary electron motion from a source placed in a region of small binding energies.

Integrating the equation thus obtained by parts, we find the equation for the average binding energy of the electron

$$\langle E \rangle = \int E f d\Gamma / \int f d\Gamma.$$

This equation has the form

$$\frac{dE}{dt} = \frac{D}{kT} \left(1 + \text{const } \frac{kT}{E} \right) \quad (\text{const} \sim 1)$$

(here the averaging brackets $\langle \rangle$ have been omitted from E).

In subsequent estimates we shall omit the second term on the right hand side of the equation, since $E > kT$. Then with the help of (3.2), we obtain

$$\frac{dE}{dt} = \frac{D}{kT} = \frac{3}{4} \frac{E}{\tau_{ee}}, \quad (3.4)$$

$$kT < E < E' = \frac{1}{2} kT \left(\frac{2I}{kT} \right)^{1/2}.$$

In the region of unidirectional motion $E > E'$, where deactivating transitions to neighboring levels predominate, the rate of change of electron binding energy is approximately equal to

$$\frac{dE}{dt} = \beta_{n,n-1} N_e \Delta E_{n,n-1}, \quad \Delta E_{n,n-1} > kT, \quad E > E'. \quad (3.5)$$

Here $\beta_{n,n-1}$ is the rate constant of the deactivating transition $n \rightarrow n-1$ and $\beta_{n,n-1} N_e$ is the atomic transition probability (per sec).

The quantity $\beta_{n,n-1}$ was derived in [6] on the basis of the quantum-mechanical formula for the cross section of the reverse process, the exciting of an atom by collision with an electron. The diffusion coefficient in the diffusion region was derived in [4] on the basis of classical mechanics. Due to the different approaches in calculating dE/dt in the two regions, a considerable discontinuity results when the two expressions for dE/dt are extrapolated to the boundary between the regions, although this discontinuity, of course, does not actually exist. In order to overcome this difficulty, we proceed in the following manner. The classical Fokker-Planck equation in the region of closely spaced levels may be obtained from the quantum-kinetic equation for the occupation numbers of the $n - x$ states, N_n , if the usual expansion is introduced in the kinetic equation with account for the fact that

$$\left(\text{with } N_n = \frac{\Delta E_{n,n-1}}{dE} \int_{E=\text{const}} f d\Gamma = \frac{f(E) \Delta E_{n,n-1}}{A(E)} \right) \Delta E_{n,n-1} \ll E_n \text{ and } \Delta E_{n,n-1} \ll kT.$$

The diffusion coefficient obtained in such an operation turns out to be equal to

$$D_1 = \beta_{n, n-1} N_e (\Delta E_{n, n-1})^2$$

and has an obvious physical meaning.

It is worthy of note that for $\Delta E_{n, n-1} \ll kT$ ($n \gg 1$) the quantum-mechanical diffusion coefficient D_1 coincides with the classical diffusion coefficient D from formula (3.2) with an accuracy to a numerical factor (D_1 turns out to be several times larger than D). In the case where $\Delta E_{n, n-1} \gg kT$ but is still in the region of large quantum numbers, the general expression for $\beta_{n, n-1}$ derived in [6] may be represented in the form

$$(\beta_{n, n-1})_{\Delta E \ll kT} \approx (\beta_{n, n-1})_{\Delta E \gg kT} \frac{kT}{\Delta E_{n, n-1}}$$

Setting this limiting value in formula (3.5) and making an approximate extension of the formula to cases where $\Delta E \approx kT$, we obtain

$$\frac{dE}{dt} = (\beta_{n, n-1})_{\Delta E \approx kT} N_e kT, \text{ or } \frac{dE}{dt} = \frac{D_1 kT}{(\Delta E_{n, n-1})^2} \quad (E > E)$$

Here D_1 is the quantum-mechanical diffusion coefficient.

In order to avoid a discontinuity in the transition from one region to the other, we here insert the classical diffusion coefficient* in place of the quantum-mechanical one:

$$D = \frac{3}{4} \frac{E kT}{\tau_{ce}}$$

Noting that $(\Delta E_{n, n-1}) = 2E/n$, $n^2 = 1/E$, we find

$$\frac{dE}{dt} = \frac{3}{16} \frac{T}{\tau_{ce}} \left(\frac{kT}{E}\right)^2 \quad (E > E)$$

Here we insert $1/\tau_{ee}$ from formula (3.3) and substitute $e^2 = 2I\alpha_0$. Finally, we obtain

$$\frac{dE}{dt} = \frac{1}{3} N_e \langle v_e \rangle \pi \left(\frac{a_0 I}{E}\right)^2 I, \quad \langle v_e \rangle = \left(\frac{8kT}{\pi m_e}\right)^{1/2} \quad (E > E). \quad (3.6)$$

Here $a_0 I/E$ is the radius of the orbit for which the binding energy is equal to E .

4. Radiative transitions and heat release. In the classical approximation the probability of the spontaneous radiative transition $n \rightarrow n'$ is equal to [8]

$$A_{n, n'} = \frac{8\pi^3 a_0^2 I^2}{m_e c^3 h^2} \frac{32}{3\pi} \frac{1}{\sqrt{3}} \frac{1}{n^3 (a')^3} \left(\frac{1}{(n')^2} - \frac{1}{n^2}\right)^{-1}. \quad (4.1)$$

Comparison with exact values shows that this formula gives a fair accuracy for any n, n' .

*Such a procedure leads to a dE/dt , i.e., the probability of deactivating transitions, several times smaller compared with [6]. This circumstance can only be favorable, since the formula [6] for $\beta_{n, n-1}$ gives values that are too high: If the formula for the cross section for excitation by electron collisions, on which the calculation of $\beta_{n, n-1}$ is based, is applied to the $1 \rightarrow 2$ transition in the hydrogen atom, then values are obtained which are roughly three times as large as the experimental values.

The probability of transitions to levels $n' \ll n$ is approximately equal to

$$A_{n, n' \ll n} \approx \frac{1.6 \cdot 10^{10}}{n^3 n'} = \frac{A_{n1}}{n'} \text{ sec}^{-1} \quad (4.2)$$

and among such transitions the most probable is the transition to the ground state $n' = 1$.

The transition probability to neighboring levels $n' = n - \Delta n$ ($1 \ll \Delta n \ll n$) is approximately equal to

$$A_{n, n-\Delta n} \approx \frac{0.8 \cdot 10^{10}}{n^3 \Delta n} \approx \frac{A_{n, n-1}}{\Delta n} \text{ sec}^{-1}. \quad (4.3)$$

Among such transitions the most probable is the transition to the closest level (we note that $A_{n1} = 2A_{n, n-1}$). As regards transitions to levels intermediate between the nearest and the ground level, for $n \gg 1$ they are of relatively small probability

$$A_{min} = A_{n, n-1} \sqrt{3} \approx A_{n, 1} \frac{2.6}{n}$$

Radiative transitions form a large number of complicated cascades, and a strict determination of the heat release E^* is possible only if calculations of the populations of atomic levels on recombination are made to take into account collisional as well as all possible radiative transitions. We shall make an estimate of E^* considering approximately the two most characteristic ways. (Recombination taking into account cascade collisional and radiative transitions was investigated numerically in [5], where, however, no data is given as regards occupation numbers.)

To begin with, we compare the rate of collisional descent (dE/dt) and the rate of "gradual" radiative descent (dE/dt), corresponding to transitions to neighboring levels. For $n \gg 1$ the latter process may be treated on the basis of classical electrodynamics, which gives for the rate of radiation of an electron revolving around an ion in a circular orbit

$$\left(\frac{dE}{dt}\right)^* = \frac{32}{3} \frac{E^3}{m_e^2 c^3 e^2} = 2.2 \cdot 10^{10} I \left(\frac{E}{I}\right)^4 \text{ energy/sec.} \quad (4.4)$$

This quantity is $\pi \sqrt{3}/4$ times larger than the quantum radiation rate corresponding to the nearest level only $h\nu_{n, n-1} A_{n, n-1}$.

The rate of gradual radiative descent (4.4), which rapidly increases as the binding energy becomes larger, reaches the rate of collisional descent (3.3), (3.4) for a binding energy

$$E^* = I \times \begin{cases} 4.3 \cdot 10^{-4} N_e^{1/2} (T)^{1/2} & (kT < E^* < E) \\ 3.1 \cdot 10^{-4} N_e^{1/2} (T^0)^{1/2} & (E^* > E) \end{cases} \quad (4.5)$$

$$E^* = I \cdot 2.1 \cdot 10^{-4} (T^0)^{1/2}$$

We shall consider the second path. In the process of collisional deactivation there always exists the probability of "jumpwise" radiative transitions to the ground level (or to the first excited level with subsequent transition to the ground level), the atom then leaving the game. The summed probability of a "jump" over all time, up to a certain moment t^* from the start of deactivation, attains a considerable magnitude, of the order of unity. The binding energy E_1^*

attained up to this moment would represent the effective heat release, if it were not for the existence of gradual radiative deactivation. Clearly, the quantity E_1^* may be determined from the approximate condition

$$\int_0^{t^*} A_{n1} dt = \int_{kT}^{E_1^*} A_{n1} \frac{dE}{(dE/dt)} \approx 1.$$

With the help of (4.2), (3.4), and (3.6), we find

$$E_1^* = I \times \begin{cases} 1.5 \cdot 10^{-4} N_e^{1/2} (T^0)^{-1/2} & (E^* < E') \\ 2.1 \cdot 10^{-4} N_e^{3/4} (T^0)^{1/4} \left[1 - 1.3 \frac{(T^0)^{1/2}}{N_e} \right]^{1/2} & (E^* > E'). \end{cases}$$

The quantities E_1^* and E^* turn out to be close to each other within a wide range of T and N_e , and in the majority of cases $E^* < E_1^*$, so that we may take the quantity (4.5), which corresponds to gradual radiation, as the heat release E^* . Formula (4.5) must be applied cautiously if the quantity E^* turns out to be large, say of the order of $E_2 = I/4$ or larger, since all the initial formulas, generally speaking, were for the case of large quantum numbers. It is better to make an immediate estimate of the transition probabilities from the $n = 2$ level to the ground level

$$A_{21} = 0.66 \cdot 10^9 \text{ sec}^{-1}, \quad \beta_{21} N_e = N_e \langle v_e \rangle \sigma_{21} \text{ sec}^{-1}.$$

The experimental cross section for deactivation by electron collision is $\sigma_{21} \sim 10^{-17} \text{ cm}^2$. Radiative transitions $2 \rightarrow 1$ predominate if $N_e < 10^{20} (T^0)^{-1/2} \text{ cm}^{-3}$.

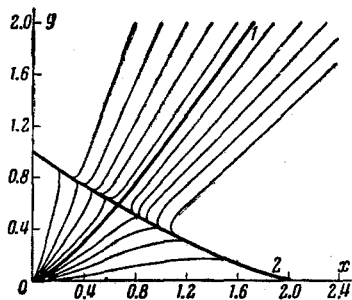


Fig. 1

5. Recombination kinetics. We will take the plasma state parameters at the moment of breakdown of ionization equilibrium as the initial conditions for the integration of the system of recombination kinetics and energy equations. This moment t_1 and the corresponding values of temperature T_1 , density N_1 , and degree of ionization α_1 may be evaluated approximately by the method of [1] with the help of the equation*

$$\left(\frac{d\alpha_p}{dt} \right)_{t=t_1} = \beta (T_1) N_1^2 \alpha_{p1}^3, \quad (5.1)$$

*At the moment of breakdown of ionization equilibrium the electron and ion temperatures are, as a rule, still equal.

taking into account that α_1 is close to the corresponding equilibrium degree of ionization α_{p1} , which is expressed in terms of T_1 and N_1 by Saha's formula. For $\alpha_{p1} \ll 1$ equation (5.1) gives approximately

$$\beta (T_1) N_1^2 \alpha_{p1}^2 t_1 = \frac{3}{2} (\gamma_1 - 1) \frac{I}{kT_1} \quad (5.2)$$

$$\left(N_1 = \frac{N^*}{\sqrt[3]{3\pi} (ut_1)^3} \right).$$

Here u is the rate of expansion of the gas cloud, N_1 is the density of atoms, and N^* is the total number of parent atoms in the cloud [8]. The relation of temperature and density, which follows from the regularity of the equilibrium stage of the expansion, must also be added to equation (5.2). Thus, if the expansion were adiabatic and the gas entropy S were equal to S_0 , then $S(N_1, T_1) = S_0$. The values N_1, T_1 may be directly related to the initial parameters of the cloud N_0, T_0 with the help of the effective adiabatic exponent γ [8]; $T_1/T_0 = (N_1/N_0)^{(\gamma-1)}$. The quantity $\gamma_1 - 1$ in formula (5.1) characterizes the temperature variation as dependent on density or time at the moment of breakdown of equilibrium:

$$\gamma_1 - 1 = \left(\frac{d \ln T}{d \ln N} \right)_{t_1} = - \frac{1}{3} \left(\frac{d \ln T}{d \ln t} \right)_{t_1}$$

(generally speaking it differs from $\gamma_* - 1$ and is determined by the derivative $(d\alpha_p/dt)_{t_1}$).

Once the initial conditions have been determined, we may proceed to integrate the system of equations in the nonequilibrium stage. Usually, the energy exchange between electrons and ions proceeds rapidly for an extended period of time after breakdown of ionization equilibrium, and the electron and heavy-particle temperatures remain close. We shall thus consider the system of equations (2.2) and (2.6), which describes just this case. An analysis of the system of equations in which E^* is given by the composite formula (4.5) would be very difficult. However, it is sufficient in practice to investigate the case where $E^* > E'$. It will be clear from what follows that if $E^*(t_1) > E'(t_1)$, then this inequality will hold for $t > t_1$ also. However, if $E^*(t_1) < E'(t_1)$, then as a result of the comparatively slow change of E^* , the latter inequality will change sign soon after the moment $t = t_1$; and so (again as a result of the slow of variation of E^*) the exact dependence of E^* is not important in the small period of time from t_1 to the moment when the inequality changes sign. Thus, we shall make further use of the second expression of (4.5) only.

Usually, E^* is fairly large compared with kT , and the degree of ionization in the nonequilibrium stage is considerably less than unity. Thus, equation (2.6) may be simplified somewhat by neglecting the corresponding quantities therein,

$$\frac{dT}{dt} + 2 \frac{T}{t} = \frac{2}{3} \frac{E^*}{k} \left(- \frac{d\alpha}{dt} \right). \quad (5.3)$$

The system (2.2) and (5.3) of two nonlinear equations in $\alpha(t)$ and $T(t)$ may be reduced in order. To do this we must pass to the new variables

$$x = - \frac{d \ln \alpha}{d \ln t}, \quad y = \frac{2}{3} \alpha \frac{E^*}{kT} x. \quad (5.4)$$

The meaning of the variable y is evident from equation (5.3), which gives

$$\frac{d \ln T}{d \ln t} = -(2 - y). \quad (5.5)$$

This means that $(2 - y)$ corresponds to the instantaneous value of the exponent m when the temperature at each moment is approximated by the power law $T \sim t^{-m}$. The quantity y gives an idea of the part played by heat release in comparison with the work done in expansion. We take the log of expressions (5.4) for x and y and differentiate with respect to $\ln t$, taking account of equation (2.2) and the dependence of $N \sim t^{-3}$ and also allowing for (5.5). Eliminating $d \ln t$ from the pair of equations thus obtained, we arrive at the basic equation

$$\frac{dy}{dx} = \frac{y(65y + 38x - 64)}{x(6(9y + 4x - 8))}. \quad (5.6)$$

This equation has four singular points (x, y) : a saddle point $(28/41, 24/41)$ and three nodes $(0, 64/65)$, $(2, 0)$, $(0, 0)$. The slopes of the separatrices at the saddle-type point are $(dy/dx)_1 = 1.12$, $(dy/dx)_2 = -0.537$. The field of integral curves is shown in Fig. 1, from which it is clear that depending on the initial conditions two types of solution are possible. If the initial point is situated to the left of the separatrix 1 (with positive slope), then the integral curves converge at the point $x = 0$, $y = 64/65$. The asymptotic laws of degree of ionization and temperature variation $\alpha \rightarrow \text{const}$, $T \sim t^{-(2-y)} = t^{-66/65}$ correspond to this point, i.e., in this case recombination ceases after a certain time. However, if the initial point is situated to the right of the separatrix 1, then the integral curves converge to the node $x = 2$, $y = 0$, which corresponds to the asymptotic laws $\alpha \sim t^{-x} = t^{-2}$, $T \sim t^{-2}$, i.e., in this case the degree of ionization always continues to diminish. The adiabatic exponent $\gamma = 5/3$ corresponds to the law $T \sim t^{-2}$. The asymptotic solution $\alpha \sim (\ln t)^{-1/2}$ corresponds to the separatrix.

The physical causes of the behavior of the degree of ionization may be easily understood in both cases. The separatrix 1 dividing the initial points of both types may be roughly approximated by the straight line $y = x$ (see Fig. 1). If the initial point lies to the left of the separatrix, then at the initial moment $y_1 > x_1$ and in agreement with (5.4)

$$\left(\frac{2}{3} \alpha \frac{E^*}{kT}\right)_1 > 1.$$

This condition means that there is in the gas a store of potential energy, significant compared with the heat energy, which passes into heat on recombination. Heat release slows down the cooling of the gas associated with the work performed in expansion, and this strongly diminishes the recombination rate, which is very sensitive to temperature.

On the other hand, if the initial point lies to the right of the separatrix, then

$$y_1 < x_1, \quad \left(\frac{2}{3} \alpha \frac{E^*}{kT}\right)_1 < 1,$$

i.e., the store of potential energy which goes to heat

the gas is small and the gas cools rapidly, which leads to more rapid recombination.

The first mode (with residual ionization, i.e., with quenching) comes about if equilibrium breaks down at high temperatures and degrees of ionization, i.e., in cases of very rapid expansion of the gas cloud (small mass of gas, high expansion velocities). In practice such a mode is possible only for $\alpha_1 \geq 1$.

The second mode, when recombination continues all the time and the degree of ionization decreases, relates to relatively slow expansion (large masses, small expansion velocities). The results of numerical integration of the equations for two typical cases are shown in Fig. 2.

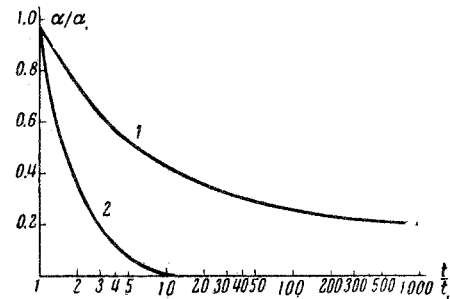


Fig. 2

Figure 2 illustrates the dependence of the degree of ionization on time. Curve 1 corresponds to the initial conditions $x_1 = 0.9$, $y_1 = 1.52$; curve 2 to conditions $x_1 = 1.72$, $y_1 = 1.77$. Given values of x_1 , y_1 may refer to very different combinations of gas parameters. For example, curve 1 describes the following concrete case: $t_1 = 2.0 \cdot 10^{-6}$ sec, $T_1 = 12\ 000^\circ$ K, $N_1 = 1.7 \cdot 10^{16}$ cm $^{-3}$, $\alpha_1 = 0.58$, $(E^*)_1 = 0.33$ I. The radius of the gas sphere $r_1 = 4.9$ cm, the expansion velocity $u = 24$ km/sec. Initial conditions for the gas cloud $r_0 = 1$ cm, $N_0 = 2 \cdot 10^{18}$ cm $^{-3}$, $T_0 = 50\ 000^\circ$ K. The ionization potential $I = 13.5$ eV, the atomic weight $A = 14$. Curve 2 describes the case $t_1 = 2.1 \cdot 10^{-6}$ sec, $T_1 = 11\ 700^\circ$ K, $N_1 = 4 \cdot 10^{16}$ cm $^{-3}$, $\alpha_1 = 0.34$, $(E^*)_1 = 0.33$ I, $r_1 = 5$ cm, $u = 24$ km/sec, $r_0 = 1$ cm, $N_0 = 5 \cdot 10^{18}$ cm $^{-3}$, $T = 50\ 000^\circ$ K, $I = 13.5$ eV, $A = 14$.

It must be remarked that in the second mode, when the degree of ionization would seem to tend to zero, further recombination stops at a certain low degree of ionization. This takes place when the energy exchange between electrons and ions breaks down and the heat liberated in recombination remains wholly in the electron gas, as a result of which the electron gas cools down more slowly. Equation (2.6) now no longer holds, and the process is described by equation (2.7). It may be shown that now only the first mode occurs, i.e., when the energy exchange between electrons and ions ceases, recombination also ceases.

In conclusion, we note once again that all the calculations are made on the assumption that the energy which is converted initially into radiation in spontaneous radiative transitions in excited atoms passes entirely out of the gas cloud and is lost to the gas. Actually, thanks to the diffusion of resonance radiation,

this energy is retained in the gas to some extent in the form of excitation energy, which enhances the role of collisions of the second type and heat release in the electron gas. This leads to a certain slowing down of cooling and recombination, i.e., the degree of ionization calculated above $\alpha(t)$ represents the lower limit of this quantity. The question of resonance radiation diffusion depends essentially on the dimensions and density of the gas cloud and the character of spectral line broadening, which determines the "depth" of the absorption line, i.e., the path length of the resonance radiation. This question should be investigated with reference to concrete conditions.

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